Generalization of Church-Rosser strategies for confluent abstract rewriting systems of the smallest uncountable cardinality

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Main assumption

- We will assume that a **background theory** for the definitions and results described below is **ZFC**
- Most of the facts given below were checked in **Isabelle/HOL** (HOL proofs can be translated to ZFC)

1. Motivation

Recall: abstract rewriting systems

Definition

An **abstract rewriting system** (**ARS**) is a pair $(X, \to)$, where

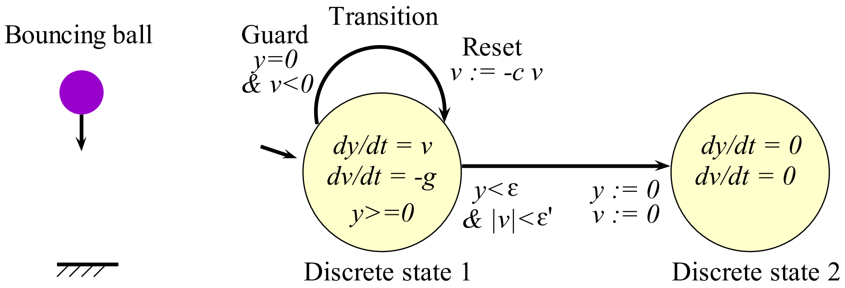
- X is a set (of ***elements***)
- $\to$ is a binary relation on X (***reduction***)

Definition

An ARS $(X, \to)$ is

1. **countable**, if X is (at most) **countable**,
   i.e. there exists a surjective function $f : \mathbb{N} \to X$.
2. **uncountable**, if X is ***not*** **countable**.

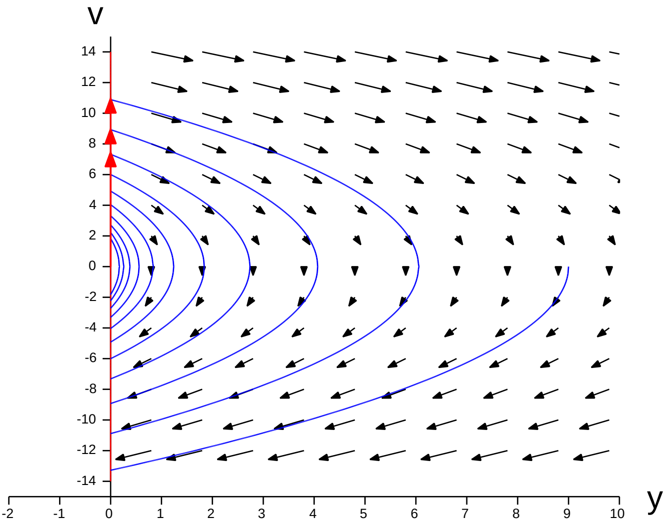
Example: uncountable ARS from hybrid system (1/3)



Define an ARS $(S, \to)$ such that

- $S = [0, +\infty) \times \mathbb{R}$ **(continuous state space)**
- $(y_1, v_1) \to (y_2, v_2)$, if (y_2, v_2) can be **reached** from (y_1, v_1)
  - **either** via **continuous evolution** within one discrete state,
  - **or** as a result of a **single discrete transition** between discrete states (that may coincide)

Example: continuous state space (2/3)



The problem of confluence of uncountable ARS

The present work was motivated in part by the following problem posed by **J. Endrullis, J.W. Klop, R. Overbeek** (color highlighting is ***not*** present in the original text)

- "*For us the most fundamental open problem is the following. As we have seen for countable systems, the question of confluence can always be reduced to local confluence. This means that every confluence diagram can always be fully tiled by elementary local confluence diagrams. For uncountable systems this question is wide open. It is conceivable that there exist complicated uncountable systems whose confluence is due to quite other properties than local confluence. ...* "[1]

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[1] J. Endrullis, J.W. Klop, R. Overbeek. *Decreasing diagrams for confluence and commutation*. Logical Methods in Computer Science, vol. 16, pp. 23:1-23:25, 2020 [*page 23*]

Why the problem of confluence of uncountable ARS is interesting and non-trivial ?

We will give 2 facts that can serve as **indirect indicators**:

1. **Set-theoretic issues** concerning uncountable directed sets
2. **Sensitivity** of generalizations of Newman’s lemma to **cardinality** / **cofinality** of an ARS

1. Set-theoretic issues

- **Confluent** ARS are closely related to **directed sets**
- Two directed sets are **cofinally similar**[2], if they can be embedded in one poset as cofinal subsets.
- **Cofinal types** are equivalence classes w.r.t. cofinal similarity
- There are **2 cofinal types** of **countable** directed sets
- For directed sets of cardinality $\leq \aleph_1$ the **number of cofinal types** depends on which axioms beyond ZFC are assumed [3]

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[2] J.W. Tukey. *Convergence and uniformity in topology*, Ann. of Math. Studies, no. 2, Princeton Univ. Press, 1940

[3] S. Todorcevic. *Directed sets and cofinal types*. Trans. of the AMS 290, Num. 2, 1985

2. Newman’s lemma

A **terminating** ARS is **confluent** if(f) it is **locally confluent**

Confluence

$$\forall a, b, c \in A \; (a \to^* b \wedge a \to^* c \Rightarrow \exists d \in A \; (b \to^* d \wedge c \to^* d))$$

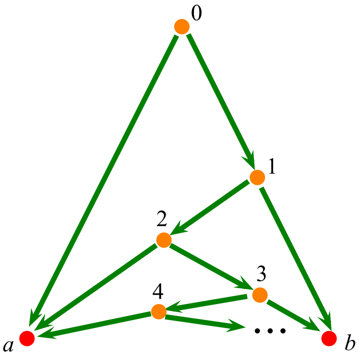
$$\begin{tikzcd}
a \arrow[r, "*"] \arrow[d, "*"'] & b \arrow[d, dashed, "*"] \\
c \arrow[r, dashed, "*"] & d
\end{tikzcd}$$

Local confluence

$$\forall a, b, c \in A \; (a \to b \wedge a \to c \Rightarrow \exists d \in A \; (b \to^* d \wedge c \to^* d))$$

$$\begin{tikzcd}
a \arrow[r] \arrow[d] & b \arrow[d, dashed, "*"] \\
c \arrow[r, dashed, "*"] & d
\end{tikzcd}$$

Known counterexamples[4] to simple generalizations

Hindley's counterexample (**with cycle**, locally confluent, *not* confluent)	**Newman's counterexample** (**acyclic**, locally confluent, *not* confluent)
	

●	reducible elements	→	reductions
●	irreducible elements	...	infinite continuation

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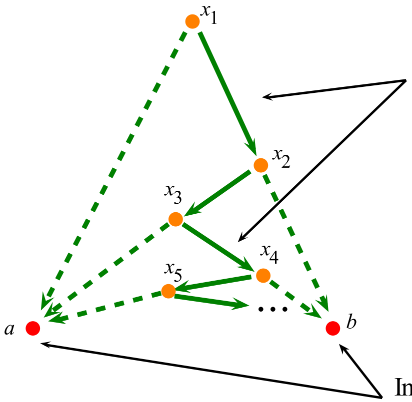
[4] G. Huet. *Confluent reductions: Abstract properties and applications to term rewriting systems*, JACM, 1980

Newman's counterexample in order-theoretic terms

- In the (pre)ordered set $(X, \to^*)$ there is an infinite **chain** with **incomparable minimal upper bounds**.
- **Local confluence** is **not sufficient** to guarantee that they have a **common upper bound**.



Given chain $\{x_i|\ i=1,2,...\}$
without greatest element
in the sense of the (pre)order ->*

Incomparable minimal upper bounds
of the chain

Extension of Newman's lemma for countable ARS

Proposition (1)

Let $(X, \rightarrow)$ *be an* ***acyclic*** *ARS such that* $(X, \rightarrow^*)$ *is a* ***cpo***[5]*.*
If X *is* ***countable****, then the following conditions are* ***equivalent****:*

1. $(X, \rightarrow)$ *is* ***confluent***
2. $(X, \rightarrow)$ *is* ***locally confluent***
3. *poset* $(X, \rightarrow^*)$ *satisfies* ***order-theoretic local confluence condition****: for every* $a \in X$ *the set* $\{x \mid a \rightarrow^* x \wedge x \neq a\}$ *has an* ***upwards linked***[6] *(in* X*)* ***coinitial subset****.*

- **Machine-checked proof** in Isabelle:
  https://pastebin.com/uLK9jqYF (temp. link, lines 136-159)
- But **without countability** assumption, conditions 1-3 are **inequivalent**.

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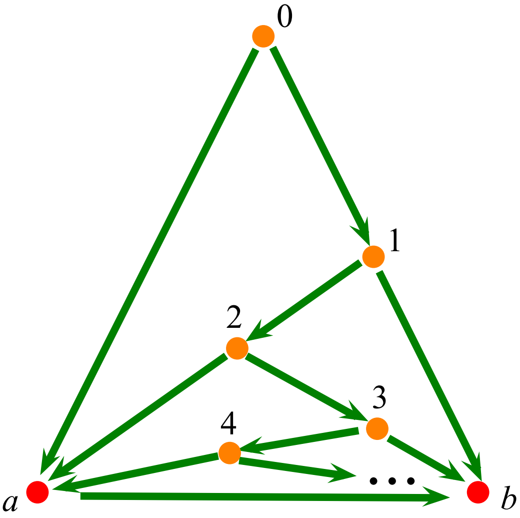
[5] a *poset* where every *direct set* has a supremum, or, alternatively, every *non-empty chain* has a supremum

[6] every *two elements* have a common upper bound *in the poset*

ARS that satisfies conditions of Proposition (1)



Spacetime ARS[7] (shows $1 \not\Leftrightarrow 3$ and $2 \not\Leftrightarrow 3$)

An **ARS** $(E, \rightarrow)$, where

- $E = \{(x,t) \in \mathbb{R} \times \mathbb{R} \mid t \leq 0\}$
- $(x,t) \rightarrow (x',t') \Leftrightarrow (t'-t) > 0 \wedge c^2(t'-t)^2 - (x'-x)^2 \geq 0.$

where:

- x, t are **space** and **time** coordinates
- $c = 1$
- E is a "**region of spacetime**"
- $\rightarrow$ is the **strict causal precedence** between events in (1+1)-dimensional **Minkowski spacetime**, **restricted** to E

Informally, it describes possibilities of **propagation of information** (e.g. using **light**) in E.

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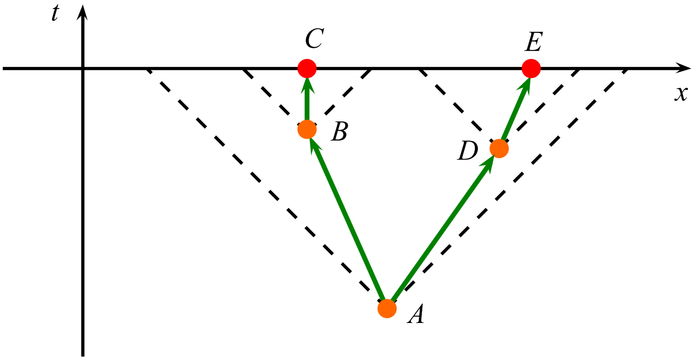
[7] I. Ivanov. *Generalized Newman's lemma for discrete and continuous systems*. LIPIcs, vol. 260, pp. 9:1-9:17, 2023.

Spacetime ARS (shows $1 \not\Leftrightarrow 3$ and $2 \not\Leftrightarrow 3$)



●	examples of reducible elements
●	examples of irreducible elements
⟶	examples of reductions
– – –	boundaries of sets of direct successors

Strengthened Newman's counterexample [8] ($1 \not\Leftrightarrow 2$)

An **ARS** $(P, \rightarrow)$, where

$$P = \mathbb{R} \times \mathbb{R} \times (\omega + 1)$$

and

$$(x, y, n) \rightarrow (x', y', n')$$

if and only if ***one** of* the following two conditions holds:

- $n < \omega \ \wedge \ n' = n + 1 \ \wedge \ |x' - x| < \frac{y'}{2^{n'}} \ \wedge \ 0 < y' < y$
- $n < \omega \ \wedge \ n' = \omega \ \wedge \ x' = x \ \wedge \ 0 = y' < y$.

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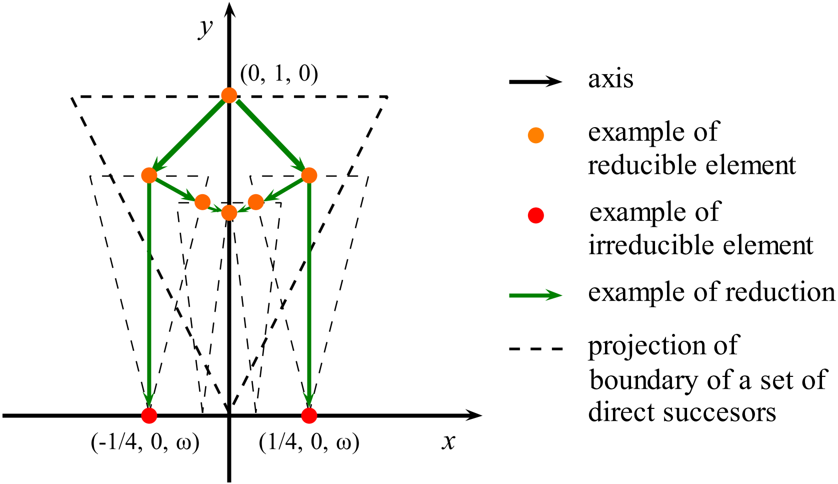
[8] I. Ivanov. *On Newman's Lemma and Non-termination*. CEUR-WS.org, vol. 3624, pp. 14-24, 2024

Strengthened Newman's counterexample [9] ($1 \not\Leftrightarrow 2$)



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[9] I. Ivanov. *On Newman's Lemma and Non-termination*. CEUR-WS.org, vol. 3624, pp. 14-24, 2024

3. Goals of this work

Goals of this work

1. **Make progress** on the problem of confluence of uncountable ARS

2. Show that **despite set-theoretic issues**, in a certain technical sense, confluent ARS of the **cardinality** $\aleph_1$ **behave closely to countable ones** and the respective diagrams can be tiled by local confluence diagrams.

3. Under the **Continuum Hypothesis** (**CH**) from 1, 2 obtain information about the behavior of **arbitrary** confluent ARS of the **cardinality of the continuum**, e.g. ARS on
    - real numbers, vectors, matrices
    - continuous real functions
    - infinite graphs on a fixed countable set of vertices
    - etc.

3. Main results

Preliminaries

- A **1-step strategy** for $(X, \to)$ is a subrelation $\to_s \subseteq \to$ such that $(X, \to_s)$ has the **same set of normal forms** as $(X, \to)$.
- For a 1-step strategy $\to_s$ the **branching weight function** $bw_{\to_s} : (\to_s) \to \mathbb{N}_0$ is defined as follows:

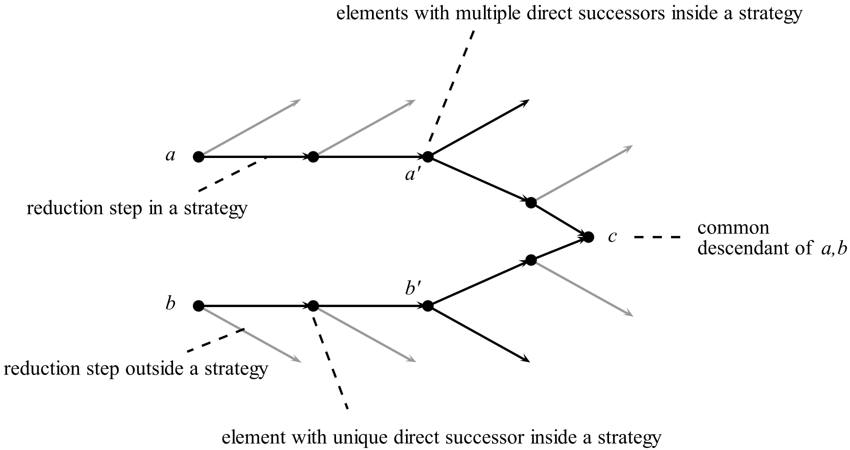
$$bw_{\to_s}((a,b)) = \begin{cases} 0, & \text{if } \{b' \mid a \to_s b'\} = \{b\}, \\ 1, & \text{if } \{b' \mid a \to_s b'\} \neq \{b\}, \end{cases}$$

- For $k \in \mathbb{N}_0$, a k**-branching 1-step nondeterministic Church-Rosser** (or k-**branching 1-nCR**) strategy for an ARS $(X, \to)$ is a 1-step strategy $\to_s$ such that

$$\forall a, b \in X \ (a \leftrightarrow^* b \Leftrightarrow B_k^{\to_s}(a) \cap B_k^{\to_s}(b) \neq \emptyset \ ),$$

where $B_k^{\to_s}(x)$ denotes the ***ball* of radius** k in $(X, \to_s)$ at an element x w.r.t. the weight function $bw_{\to_s}$.

k-branching 1-nCR strategy



Every pair of **convertible elements in ARS** (e.g. a, b) can be joined using reduction sequences in the strategy that **pass through no more than k branching points** in the strategy.

Main theorem

Theorem (no. 1 in the paper)

Let $(X, \to)$ *be an ARS such that* $|\to| \leq \aleph_1$*.*
Then the following conditions are equivalent:

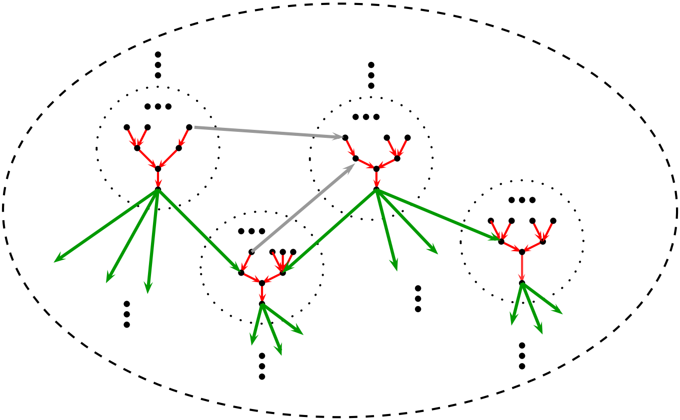
- $(X, \to)$ *is* ***confluent***
- $(X, \to)$ ***has*** *a* 1-***branching 1-nCR strategy****,*
  i.e. a strategy $\to_s$ *such that*

$$\forall a, b \in X \ (a \leftrightarrow^* b \ \Leftrightarrow \ B_1^{\to_s}(a) \cap B_1^{\to_s}(b) \neq \emptyset \ )$$

Formal proof in Isabelle:

https://doi.org/10.5281/zenodo.18079151 (lines 2693-2697)

Decomposition of confluent relation of cardinality $\aleph_1$



Red and **green** arrows form a **1-branching 1-nCR strategy**.
Grey arrows are reduction steps outside of this strategy.
Red arrows are **deterministic steps** within the strategy,
green arrows are **non-deterministic** steps within the strategy.

A 1-step **nondeterministic local Church-Rosser strategy** (**1-nLCR strategy**) in $(X, \to)$ is a 1-step strategy $\to_s$ such that

$$\forall a, b \in X \; (\; a \to b \wedge a \to c \;\; \Rightarrow \;\; B_1^{\to_s}(b) \cap B_1^{\to_s}(c) \neq \emptyset \;)$$

Proposition

If $| \to | \leq \aleph_1$, then the following 2 conditions are equivalent:

1. $(X, \to)$ *is* ***confluent***
2. $(X, \to)$ *has a* ***1-nLCR strategy***

Follows from Theorem 1 and formal theory:

https://pastebin.com/pbFCNNzR (temp. link, lines 1812-1852)

Proposition

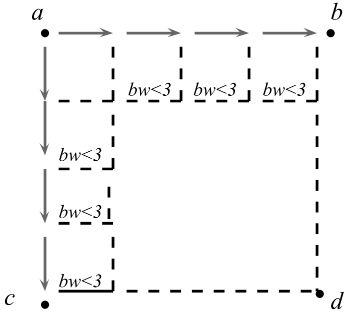
Let $\to_s$ be a ***1-nLCR strategy*** *in $(X, \to)$. Then for every $k \in \mathbb{N}$:*

$$\forall a, b \in X \;\; (a \leftrightarrow^k b \;\; \Rightarrow \;\; B_{2k}^{\to_s}(a) \cap B_{2k}^{\to_s}(b) \neq \emptyset \;\; \Rightarrow \;\; a \leftrightarrow^* b \;)$$

https://pastebin.com/pbFCNNzR (temp. link, lines 1854-1872)

Completing confluence diagrams with 1-nLCR strategy

reduction sequences from a to b and a to c of length k (possibly outside a strategy)



reduction sequence from b to d inside a strategy
that has branching weight not exceeding 2k

reduction sequence from c to d inside a strategy
that has branching weight not exceeding 2k

Example 1

Let $(X, \rightarrow)$ be an ARS where $X = \mathbb{R}^2$.

$$(x, y) \rightarrow (ky, kx) \text{ for every } k \in \mathbb{R} \text{ such that } k > 0,$$

Consider $\rightarrow_s \subseteq X \times X$ such that

$$(x, y) \rightarrow_s (x', y')$$

if and only if the following conditions hold:

1. $(x, y) \rightarrow (x', y')$
2. if $x > y$, then $(x', y') = (y, x)$.

Example 2

Let $(X, \to)$ be ARS where X is the set of all **finite** subsets of $\mathbb{R}$ (i.e. elements of the ARS are finite sets of real numbers) and

$$\to = \{(a, b) \in X \times X \mid \exists x \in \mathbb{R}\ b = a \cup \{x\}\}.$$

Then $(X, \to)$ is **confluent**, so under **Continuum Hypothesis** it has a **1-branching 1-nCR strategy.**
(in the given case it can also be obtained explicitly without CH).

Idea of the proof of the main theorem

Let $U \neq \emptyset$ be a **fixed set** and k be a **positive integer**.

- 2^{U^k} denotes the set of all k-**ary relations** on U
- $\mathcal{P}_f(A)$ denotes the set of all **finite subsets** of a set A

Definition

A set $\mathcal{R} \subseteq 2^{U^k}$ of k-ary **relations** on U

- has **finite character**, if for every $A \in 2^{U^k}$,

$$A \in \mathcal{R} \;\Leftrightarrow\; \mathcal{P}_f(A) \subseteq \mathcal{R}$$

- is **characterized by finite extensions**, if for every $A \in 2^{U^k}$,

  $A \in \mathcal{R} \;\Leftrightarrow\; \mathcal{P}_f(A) \cap \mathcal{R}$ is a **cofinal subset** of $\mathcal{P}_f(A)$ w.r.t. $\subseteq$

  i.e.

$$A \in \mathcal{R} \;\Leftrightarrow\; (\forall F \in \mathcal{P}_f(A)\;\; \exists F' \in \mathcal{P}_f(A) \cap \mathcal{R}\quad F \subseteq F')$$

Definition

- A set $\mathcal{R} \subseteq 2^{U^k}$ is **closed under chain unions**, if for every $\mathcal{C} \subseteq \mathcal{R}$:

$$(\forall A, B \in \mathcal{C} \;\; A \subseteq B \vee B \subseteq A) \;\Rightarrow\; \left(\bigcup \mathcal{C}\right) \in \mathcal{R}$$

- A set $\mathcal{R}' \subseteq 2^{U^k}$ is the **chain-union closure** of $\mathcal{R} \subseteq 2^{U^k}$, if $\mathcal{R}'$ is the **least** (in the sense of $\subseteq$) subset of 2^{U^k} such that

$\mathcal{R} \subseteq \mathcal{R}'$ and $\mathcal{R}'$ is **closed under chain unions**

Example. Let $k = 1$ and $U = \mathbb{N} = \{1, 2, 3, ...\}$. The set

$$\{\{1, 2, ..., k\} \mid k \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}$$

is the **chain-union closure** of

$$\{\{1, 2, ..., k\} \mid k \in \mathbb{N}\}$$

Theorem (variant of least fixed point induction)

If *a set* $\mathcal{R} \subseteq 2^{U^k}$ *of* k*-ary relations on* U

- *is* ***characterized by finite extensions***
- *and contains the* ***empty relation****, i.e.* $\emptyset \in \mathcal{R}$,

then $\mathcal{R}$ *is the* ***chain-union closure*** *of* $\{A \in \mathcal{R} \mid A \text{ is finite}\}$.

Proposition

The set $CR(U) = \{r \in 2^{U \times U} \mid (U, r) \text{ is a } \textbf{confluent} \text{ ARS}\}$ *is* ***characterized by finite extensions*** *and contains* $\emptyset$.

Machine-checked proof in Isabelle:

https://pastebin.com/UFEKVu9y
(temp. link, lines 1242-1245, 1453)

Corollary

$CR(U)$ *is the* ***chain-union closure*** *of* $\{r \in CR(U) \mid r \text{ is } \textbf{finite}\}$.

4. Conclusions

Conclusions

1. Every **confluent ARS** $(X, \to)$ of the **cardinality** $\aleph_1$ has a well-behaved confluent strategy (**1-branching 1-nCR strategy**).

2. Confluence of ARS of the cardinality $\aleph_1$ can be characterized in terms of the existence of **1-nLCR strategies** that, in a sense, generalizes Newman's lemma.

3. **Confluence** criteria for ARS of **higher cardinalities** is a subject of **further research**.